Problem Solving Through History
Problem Set #10

1. Show that if a set is countably infinite you remove one element of the set, then the set is still countably infinite. (Be sure to show the mapping.)

2. Suppose that A is a countably infinite set. Show that you can remove a countably infinite number of elements from A so that what remains is still countabilty infinite.

3. Show that $|R^1| = |R^3|$ [R^1 is the real line and R^3 is 3 –dimensional space.]

4. Show that if A is a finite set with n elements, then $|P(A)| = 2^n$.

5. Which of the functions are one-to-one and why? (You should use the definition of one-to-one to support your claims if possible.)

   a) $f(x) = 4 - 2x$
   b) $f(x) = \frac{1}{x} + x^2$
   c) $f(x) = 4 + x^4$

6. Let Q^+ be the set of all positive rational numbers (fractions).
Show that f: $Q^+ \rightarrow N$ defined by $f(\frac{m}{n}) = 2^m 3^n$ is 1 to 1 and therefore Q^+ is countably infinite.

7. Which of the following sets are countably infinite and why? [If yes, be sure to describe the mapping.]

   a) The set of all lines in the plane, each of which passes through the origin. (Hint consider the slopes of the lines.)

   b) The set of all lines in the plane, each of which passes through the origin and a point having both coordinates rational.

   c) The set of all intervals on the real line having both endpoints rational.

   d) Any infinite set of non-overlapping intervals on the real line.

8. Show that between any two rational numbers there is an irrational number, and between any two irrational numbers there is a rational number.

9. Prove or disprove: Suppose for each positive integer n, A_n is countably infinite and that $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq ...$. Then $\cap A_n \neq \varnothing$.